

# On the Axiomatics of the 5-dimensional Projective Unified Field Theory of Schmutzer

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## Abstract

For more than 40 years E.Schmutzer has developed a new approach to the (5-dimensional) projective relativistic theory which he later called Projective Unified Field Theory (PUFT). In the present paper we introduce a new axiomatics for Schmutzer's theory. By means of this axiomatics we can give a new geometrical interpretation of the physical concept of the PUFT.

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## 1 Introduction

As it is well known the 5-dimensional idea of a unified field theory goes back to the works of Kaluza and Klein [1, 2]. The pioneers of the projective approach to this theory were Veblen and van Dantzig [3, 4]. Later this approach was developed further by many other authors.

An essential progress in this projective type of theories was done by Jordan [5] who first took into consideration the occurring scalar field which inevitably appears in this theory. However, the field equations used by him were unacceptable.

A basically different approach to a projective field theory was proposed by E.Schmutzer [6] who (according to the requirements of a unified field theory) developed further and applied a basis vectors formalism initiated by Hessenberg, Schouten and others [7] in the theory of manifolds. He had no longer considered the scalar field mentioned above to be an auxiliary one. On the contrary he associated this field with a new phenomenon of nature being on one the level of gravitation and electromagnetism. In 1980 he introduced the new term “**scalarism**” [8] for this phenomenon. For this hypothetically used scalar field with fundamental importance in

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<sup>†</sup>Dedicated to my academic teacher Prof. Dr. Ernst Schmutzer on the occasion of his 70<sup>th</sup> birthday.

the PUFT Schmutzer introduced the term “**scalaric field**” in order to distinguish it from the various other scalar fields in physics. The most interesting and important results of application of PUFT are presented in the Appendix.

Beside the projective relativistic theory many authors were actively developing further the initial Kaluza - Klein theory aiming at a unified field theory of elementary particles. Here we only refer to the monographs of Wesson [9] and Vladimirov [10], where one can find references to the historical, mathematical and physical literature on this subject.

Concluding this introduction we would like to mention the new 5-dimensional original field theory by Wesson [11, 9, 12] recently appeared and offered for discussion.

In the first two versions of PUFT (see [6] and [8] respectively) Schmutzer used 5-dimensional Einstein-like field equations.<sup>1</sup>

$${}^5R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} {}^5R + \Lambda_{\mu\nu} = \kappa_0 \Theta_{\mu\nu} \quad (1)$$

Afterwards with the help of a special projection procedure (details can be found in the papers quoted above) a system of 4-dimensional field equations describing gravitation, electromagnetism and scalarism was derived. Here  $\kappa_0 = \frac{8\pi G}{c^4}$  is Einstein’s gravitational constant,  $\theta_{\alpha\varepsilon}$  is the so-called energy projector of the non-geometrized matter named “**substrate**”,  ${}^5R_{\alpha\varepsilon}$  is the 5-dimensional Ricci tensor,  ${}^5R$  is the 5-dimensional curvature invariant, and

$$\text{a) } \Lambda_{\mu\nu} = \lambda_0 g_{\mu\nu} \quad \text{resp.} \quad \text{b) } \Lambda_{\mu\nu} = \lambda_0 (g_{\mu\nu} + s_\mu s_\nu) S \quad (2)$$

are analogs of the cosmological terms in Version I and Version II, respectively. Here  $g_{\mu\nu}$  is the metric tensor. In a special frame  $\{X^\mu\}$  the unit vector  $s^\mu$  has (see the next section for details) the following form  $s^\mu = \frac{X^\mu}{S}$ , where  $S = \sqrt{X_\mu X^\mu} = S_0 e^\sigma$  ( $S_0$  is an arbitrary constant of the dimension of length).

The 5-dimensional Ricci tensor and the 5-dimensional curvature invariant, both mentioned above, are defined as follows:  ${}^5R_{\alpha\varepsilon} = {}^5R^\tau_{\alpha\varepsilon\tau}$ ,  ${}^5R = {}^5R^\alpha_\alpha$ , where

$${}^5R^\alpha_{\mu\nu\varepsilon} \equiv \left\{ \alpha \right\}_{\mu\varepsilon, \nu} - \left\{ \alpha \right\}_{\mu\nu, \varepsilon} + \left\{ \tau \right\}_{\mu\varepsilon} \left\{ \alpha \right\}_{\tau\nu} - \left\{ \tau \right\}_{\mu\nu} \left\{ \alpha \right\}_{\tau\varepsilon}. \quad (3)$$

For physical reasons in the following the Gauss system of units is chosen.

**Version I.** The 4-dimensional field equations (without a cosmological term:  $\Lambda_{\mu\nu} = 0$ ) have the following form:

$${}^4R_{mn} - \frac{1}{2}g_{mn} {}^4R = \kappa(E_{mn} + S_{mn} + \Theta_{mn}). \quad (4)$$

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<sup>1</sup>Greek indices run from 1 to 5, Latin indices from 1 to 4; the signatures are: of the 5-dimensional metric (+ + + - +), of the space-time metric (+ + + -). Comma means the partial and semicolon the covariant derivative, respectively.

These equations are generalized 4-dimensional equations for the gravitational field, where  $\kappa = \kappa_0 e^{-\sigma}$ ,

$$E_{mn} = \frac{1}{4\pi}(B_{mk}H^k_n + \frac{1}{4}g_{mn}B_{jk}H^{jk}) \quad (5)$$

is the electromagnetic energy-momentum tensor and

$$S_{mn} = -\frac{1}{2\kappa}(\sigma_{,m}\sigma_{,n} + \sigma_{,m;n} - g_{mn}(\sigma_{,k}\sigma^{,k} + \sigma^{,k}_{;k})) \quad (6)$$

is the energy-momentum tensor of the scalaric field  $\sigma$ . Further following field equations hold:

$$a) H^{mn}_{;n} = \frac{4\pi}{c}j^m, \quad b) B_{[mn,k]} = 0, \quad c) H_{mn} = e^{3\sigma}B_{mn}, \quad (7)$$

$$\sigma^{,k}_{;k} = \kappa_0 \left( \frac{2}{3}\vartheta + \frac{1}{8\pi}B_{kj}H^{kj} \right). \quad (8)$$

These are the electromagnetic field equations and the field equation for the scalaric field  $\sigma$ . Here the following notations were used:  $\overset{4}{R}_{mn}$ ,  $\overset{4}{R}$  are the Ricci tensor and the curvature invariant in the 4-dimensional space-time, respectively.  $H^{mn}$ ,  $B^{mn}$  are the electromagnetic induction tensor and the electromagnetic field strength tensor, respectively. The quantity  $\vartheta$  being one of the sources of the scalaric field is called scalaric substrate energy density.

The idea of developing the **Version II** [8] was to remove the second order derivatives in the energy – momentum tensor (6) of the scalaric field. By means of a modified projection formalism it became possible to obtain a system of equations being slightly different from the analogous system of the version I, given by the equations (6), (7c) and (8), namely:

$$\overset{4}{R}_{mn} - \frac{1}{2}\overset{4}{R}\overset{4}{g}_{mn} + \lambda_0 S_0 \overset{4}{g}_{mn} = \kappa_0(\theta_{mn} + E_{mn} + S_{mn}), \quad (9)$$

$$S_{mn} = -\frac{3}{2\kappa_0}(\sigma_{,m}\sigma_{,n} - \frac{1}{2}g_{mn}\sigma_{,k}\sigma^{,k}), \quad (10)$$

$$a) H^{mn}_{;n} = \frac{4\pi}{c}j^m, \quad b) B_{[mn,k]} = 0, \quad c) H_{mn} = e^{3\sigma}B_{mn}, \quad (11)$$

$$\sigma^{,k}_{;k} = \kappa_0 \left( \frac{2}{3}\vartheta + \frac{1}{8\pi}B_{kj}H^{kj} \right). \quad (12)$$

However, the new projection formalism led to other problems, particularly in the spinor theory. Therefore approximately in 1994 E.Schmutzer left this version II and offered version III.

In the **version III** by deeply founded considerations on the level of the Lagrange-Hamilton formalism the following new 5-dimensional field equations were found [13]:

$$\begin{aligned}
& R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} \overset{5}{R} - \frac{1}{S}S_{,\mu;\nu} + \frac{K_0\kappa_0}{S^2}S_{,\mu}S_{,\nu} - \frac{1}{S}s_\mu s_\nu [2(1 - \frac{1}{2}K_0\kappa_0)S_{;\tau}^\tau \\
& - \frac{3}{S}(1 - \frac{1}{2}K_0\kappa_0)S_{,\tau}S^{,\tau} + \frac{3\lambda_0}{S} - \frac{S}{2}\overset{5}{R}] + \frac{1}{S}g_{\mu\nu}[S_{;\tau}^\tau \\
& - \frac{1}{S}(1 + \frac{1}{2}K_0\kappa_0)S_{,\tau}S^{,\tau} + \frac{1}{S}\lambda_0] = \kappa_0\Theta_{\mu\nu}
\end{aligned} \tag{13}$$

( $\lambda_0$  is a kind of cosmological constant.  $K_0$  is a free constant, where Schmutzer preferred the choice  $K_0 = -2$ ). Compared with the Einstein-like field equation (1) this is a rather complicated equation, but it fulfils important physical demands mentioned in the [13]. From (13) one obtains

$$R_{mn} - \frac{1}{2}g_{mn}R + \frac{\lambda_0}{S_0^2}e^{-2\sigma}g_{mn} = \kappa_0(E_{mn} + S_{mn} + \Theta_{mn}), \tag{14}$$

$$E_{mn} = \frac{1}{4\pi}(B_{mk}H^k_n + \frac{1}{4}g_{mn}B_{jk}H^{jk}), \tag{15}$$

$$S_{mn} = -K_0(\sigma_{,m}\sigma_{,n} - \frac{1}{2}g_{mn}\sigma_{,k}\sigma^{,k}), \tag{16}$$

$$a) H^{mn}_{;n} = \frac{4\pi}{c}j^m, \quad b) B_{[mn,k]} = 0, \quad c) H_{mn} = e^{2\sigma}B_{mn}, \tag{17}$$

$$\sigma^{,k}_{;k} = \frac{1}{K_0}\left(\vartheta + \frac{1}{8\pi}B_{kj}H^{kj}\right) + \frac{2\lambda_0}{\kappa_0 K_0 S_0^2}e^{-2\sigma}. \tag{18}$$

In the present paper we introduce a new geometrical axiomatics for the Schmutzer theory. By means of this axiomatics we can give a new geometrical interpretation of physical results obtained in the PUFT.

## 2 Projection formalism

As it is well known, the physical basis of the 5-dimensional Projective Unified Field Theory is supported by the following mathematical theorem: The semidirect product of the Abelian group of gauge transformations (electromagnetism) and of the group of the general 4-dimensional coordinate transformations (gravitation) corresponds to the group being homomorphic to the group of all 5-dimensional homogeneous of degree 1 coordinate transformations

$$X^{\mu'} = X^{\mu'}(X^\nu) = \frac{1}{\alpha}X^{\mu'}(\alpha X^\nu) \quad (\alpha = \text{const}). \tag{19}$$

This mathematical theorem allows us to assume that the geometry, constructed on this group, can be a basis for the geometrization of the electromagnetic, the gravitational and the scalaric field.

From the equation (19) and Euler's theorem on homogeneous functions follows that these special coordinates  $X^\mu$  in the 5-dimensional space  $\mathcal{M}_5$  are transformed as the components of a vector:

$$X^{\mu'} = X^{\mu'}_{,\nu} X^\nu. \quad (20)$$

Further the vector

$$\mathbf{R} = X^\mu \mathbf{E}_\mu, \quad \mathbf{E}_\mu = \frac{\partial}{\partial X^\mu} \quad (21)$$

can be regarded as 5-dimensional radius vector. This was a very important starting point of Schmutzer in 1957. Also in the following this vector field  $\mathbf{R}$  plays a fundamental role. Of course, it is possible to introduce in the space  $\mathcal{M}_5$  arbitrary coordinates  $y^\mu = y^\mu(X^\alpha)$ .

In context with the theorem mentioned above we should remark that the 4-dimensional coordinates  $\{x^i\}$  in the space-time should satisfy the equation

$$x^i_{,\nu} X^\nu = 0 \quad (22)$$

(for details see [6]).

In order to construct a projection formalism, let us consider the congruence

$$y^\nu = y^\nu(x^i, \tau) \quad (23)$$

of integral curves of the vector field  $\mathbf{R}$ , where  $\tau$  is a continuous parameter specified along each curve ( $x^i = \text{const}$ ) of this congruence.

The congruence (23) is the starting point of our consideration. In general the quantities  $x^i$  are not the first four coordinates of a 5-dimensional coordinate system.

Hereinafter we will consider the 4-dimensional hypersurface  $\tau(y^\nu) = \text{const}$  to be the 4-dimensional space-time. Moreover, the parameter  $\tau$  should be chosen to make tangent vectors

$$\xi^\nu(x^i, \tau) \equiv \frac{\partial}{\partial \tau} y^\nu(x^i, \tau) \quad (24)$$

coinciding with the vectors  $X^\nu$ :

$$\xi^\nu(x^i, \tau) = X^\nu. \quad (25)$$

It is important to point out that the equation (25) is only valid in the frame  $\{X^\nu\}$ ; but it can always be rewritten in an arbitrary frame  $\{y^\nu\}$ :  $\xi^\nu(x^i, \tau) = \mathcal{R}^\nu$ , where  $\mathcal{R}^\nu$  are components of the vector  $\mathbf{R}$  in the coordinate basis  $\mathbf{e}_\nu = \frac{\partial}{\partial y^\nu}$ . Let us emphasize that all equations containing vectors  $X^\mu$  are only valid in the special frame. Henceforth it will not be specially accentuated.

According to (22) we postulate equality to zero of the Lie derivative with respect to  $\mathbf{R}$  for any 4-dimensional quantity (i.e. quantity which depends only on 4-dimensional coordinates).

It is quite natural to extend the introduced postulate on all 5-dimensional vectors and tensors which further will be associated with the 4-dimensional quantities :

$$\mathcal{L}\mathcal{R}\mathbf{T} = 0 \quad (\mathcal{L}\mathcal{R}T^{\alpha\dots\beta\dots} = 0). \quad (26)$$

In the coordinate basis (21) one can rewrite the last equation (26) in the form

$$T^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_m,\lambda}X^\lambda = (n-m)T^{\mu_1\dots\mu_n}_{\nu_1\dots\nu_m}. \quad (27)$$

The geometrical quantities satisfying the projector condition (27) are called projectors [3, 4]. By applying the projector condition (26) to a metric tensor  $\mathbf{g}$  we obtain

$$\mathcal{L}\mathcal{R}\mathbf{g} = 0 \quad (g_{\mu\nu,\varepsilon}X^\varepsilon = -2g_{\mu\nu}). \quad (28)$$

From the last equation follows that the 5-dimensional radius vector  $\mathcal{R}$  is a Killing vector. Thus the congruence (23) is a Killing congruence (see for example [14]).

In order to study geometrical properties of this congruence we introduce a unit vector  $\mathbf{s}$ , i.e.

$$\mathbf{s} = \frac{\mathcal{R}}{S} \quad \left( s^\mu = \frac{X^\mu}{S} \right), \quad (29)$$

where  $S = \sqrt{\mathbf{g}(\mathcal{R}, \mathcal{R})} = \sqrt{g_{\mu\nu}X^\mu X^\nu}$ . From the definition (29) it is clear that  $\mathbf{s}$  is the unit tangential vector field to the lines of the congruence (23).

In order to provide a description of geometrical properties of the congruence (23) we introduce, as usually, the following quantities:

$$a) \ G^\mu \equiv s^\nu s^\mu_{;\nu}, \quad b) \ \omega_{\mu\nu} \equiv P^\tau_\mu P^\varepsilon_\nu s_{[\tau,\varepsilon]} \quad c) \ D_{\mu\nu} \equiv P^\tau_\mu P^\varepsilon_\nu s_{(\tau;\varepsilon)}, \quad (30)$$

where

$$\begin{aligned} G^\mu &\implies \text{the first curvature vector of the} \\ &\quad \text{lines of the congruence;} \\ \omega_{\mu\nu} &\implies \text{the angular velocity tensor of the} \\ &\quad \text{congruence;} \\ D_{\mu\nu} &\implies \text{rate-of-strain tensor of the con-} \\ &\quad \text{gruence.} \end{aligned}$$

The quantity

$$P^\tau_\mu = \delta^\tau_\mu - s^\tau s_\mu \quad (31)$$

is the projection tensor. The semicolon means the Riemannian covariant derivative ( $\overset{R}{\nabla}$ ):

$$\overset{R}{\nabla}_{\mathbf{e}_\tau} \mathbf{e}_\alpha = \left\{ \begin{matrix} \varepsilon \\ \alpha\tau \end{matrix} \right\} \mathbf{e}_\varepsilon \quad \left( \mathbf{e}_\alpha = \frac{\partial}{\partial x^\alpha} \right) \quad (32)$$

with  $\left\{ \begin{smallmatrix} \varepsilon \\ \alpha\tau \end{smallmatrix} \right\} \equiv \frac{1}{2}g^{\varepsilon\sigma}(g_{\sigma\alpha,\tau} + g_{\tau\sigma,\alpha} - g_{\alpha\tau,\sigma})$ . If we take into account that the vector field  $\mathcal{R}$  is Killingian, we obtain

$$\begin{aligned} \text{a)} \quad G^\mu &= \frac{1}{2S}X^{\varepsilon\mu}s_\varepsilon = -\frac{S^{\cdot\mu}}{S}, \\ \text{b)} \quad \omega_{\mu\nu} &= \frac{1}{2S}P^\varepsilon_\mu P^\tau_\nu X_{\tau\varepsilon}, \\ \text{c)} \quad D_{\mu\nu} &= 0, \end{aligned} \tag{33}$$

where the following abbreviation was used:

$$X_{\mu\nu} = X_{\nu,\mu} - X_{\mu,\nu}. \tag{34}$$

From the equations (30) and (33) we obtain the following important relations:

$$s_{\mu;\nu} = D_{\mu\nu} + \omega_{\mu\nu} + G_\mu s_\nu, \tag{35}$$

$$\frac{1}{2S}X_{\nu\mu} = \omega_{\mu\nu} + \frac{1}{S}(s_\mu S_{,\nu} - s_\nu S_{,\mu}). \tag{36}$$

From the relation (33b) follows that in general a holonomic hypersurface orthogonal to the given congruence does not exist. (The case  $\omega_{\mu\nu} = 0$  is physically not interesting, since further the angular velocity of the congruence will be associated with the electromagnetic field). Therefore in contrast to Schmutzer's orthogonality approach of space-time (based on the basis vector formalism) here we want to offer an alternative version of this problem: we shall identify space-time with a 4-dimensional hypersurface in the 5-dimensional space abandoning the requirement of **orthogonality** of this hypersurface to the congruence.

Let us consider some hypersurface  $\tau(X^\alpha) = \text{const}$ . As far as a parameter  $\tau$  cannot univally be derived from the equations (24) and (25), then hypersurfaces  $\tau(X^\alpha) = \text{const}$  are not defined univally either. Therefore we can choose in  $\mathcal{M}_5$  an arbitrary hypersurface which we shall identify with hypersurface  $\tau(X^\alpha) = 0$ . This hypersurface should only satisfy the condition  $X^\alpha\tau_{,\alpha} \neq 0$ . With an exponential map we can extend it along the lines of congruence (23) to a finite region in  $\mathcal{M}_5$ . Thus we receive a one-parametric set of hypersurfaces. Hence from the equations (24) and (25) follows that

$$< d\tau, \boldsymbol{\xi} > = 1 \quad (\xi^\varepsilon\tau_{,\varepsilon} = X^\varepsilon\tau_{,\varepsilon} = 1) \tag{37}$$

and

$$\mathcal{L}\mathcal{R}d\tau = 0 \quad (X^\varepsilon\tau_{,\alpha,\varepsilon} + X^\varepsilon_{,\alpha}\tau_{,\varepsilon} = 0). \tag{38}$$

From the last relation we can conclude that the one-form  $d\tau$  which further we also shall denote by  $\zeta$  satisfies the projector condition (27):

$$\mathcal{L}\mathcal{R}\zeta = 0, \quad \zeta \equiv d\tau \quad (\zeta_{\mu,\tau}X^\tau = -\zeta_\mu). \tag{39}$$

The unit one-form  $\boldsymbol{\nu} = \Lambda \boldsymbol{\zeta}$  also fulfills this condition:

$$\mathcal{L}_{\boldsymbol{\mathcal{R}}} \boldsymbol{\nu} = 0 \quad (\nu_{\tau, \mu} X^\mu = -\nu_\tau), \quad (40)$$

where  $\Lambda = \langle \boldsymbol{\nu}, \boldsymbol{\mathcal{R}} \rangle = \nu_\varepsilon X^\varepsilon$  and  $\nu_\varepsilon \nu^\varepsilon = 1$ .

Above we introduced the projection tensor  $P^\alpha_\varepsilon$ . However, the hypersurface  $\tau(X^\alpha) = 0$  (we also shall denote it by  $\mathcal{M}_4$ ) is not orthogonal to the congruence (23). Therefore it is possible to define two more projection tensors:

$$\text{a) } b_{\alpha\varepsilon} \equiv g_{\alpha\varepsilon} - \nu_\alpha \nu_\varepsilon, \quad \text{b) } h^\alpha_\varepsilon \equiv g^\alpha_\varepsilon - \xi^\alpha \zeta_\varepsilon. \quad (41)$$

All these projection tensors satisfy the projector condition (27):

$$P^\alpha_{\varepsilon, \nu} X^\nu = 0, \quad h^\alpha_{\varepsilon, \nu} X^\nu = 0, \quad b_{\alpha\varepsilon, \nu} X^\nu = -2b_{\alpha\varepsilon}. \quad (42)$$

The projection tensor  $b_{\alpha\varepsilon}$  sometimes is called the first fundamental form of  $\mathcal{M}_4$  or the induced metric on  $\mathcal{M}_4$ . (In the following we shall define the induced metric on  $\mathcal{M}_4$  in a somewhat different way). The tensor  $\chi_{\alpha\varepsilon}$  defined on the hypersurface  $\tau = 0$  by

$$\chi_{\alpha\varepsilon} \equiv b^\mu_\alpha b^\nu_\varepsilon \nu_{(\mu; \nu)} = \frac{1}{2\Lambda} \mathcal{L}_{\boldsymbol{\lambda}} b_{\alpha\varepsilon}, \quad (43)$$

is called the second fundamental form or the exterior curvature of  $\tau = 0$ . Here the following abbreviations were used:

$$\lambda^\varepsilon \equiv \Lambda \nu^\varepsilon = X^\varepsilon - \mathcal{X}^\varepsilon, \quad \mathcal{X}^\varepsilon \equiv b^\varepsilon_\alpha X^\alpha. \quad (44)$$

The above introduced projection tensors in general differ from each other. Therefore the question, which of them should be used for the projection of 5-dimensional vectors and tensors into the 4-dimensional hypersurface, is not trivial. In order to give an answer to this question, we consider the map  $\phi$ :

$$\phi: \mathcal{M}_5 \xrightarrow{\phi} \mathcal{M}_4. \quad (45)$$

The map  $\phi$  should be defined in such a way to make mapped quantities not depending on the parameter  $\tau$ , i.e. on the ‘‘fifth coordinate’’ (cylinder condition). This requirement means that all points laying on the same line of the congruence are mapped to the same point on the hypersurface  $\tau(X^\nu) = \text{const}$ . The elementary map of this type is an exponential map (see Fig. 1).

The coordinates of the point  $P_1$  satisfy the relation

$$X^\alpha(P_1) = X^\alpha(x_0^m, \tau_1) = X^\alpha(x_0^m, 0) \exp(\tau_1), \quad (46)$$

where  $X^\alpha(x_0^m, 0) = X^\alpha(P')$ . Using the equations (24) and (25) one can obtain

$$\phi_\tau: X^\alpha(P) = \exp(\tau) X^\alpha(\phi_\tau(P)). \quad (47)$$



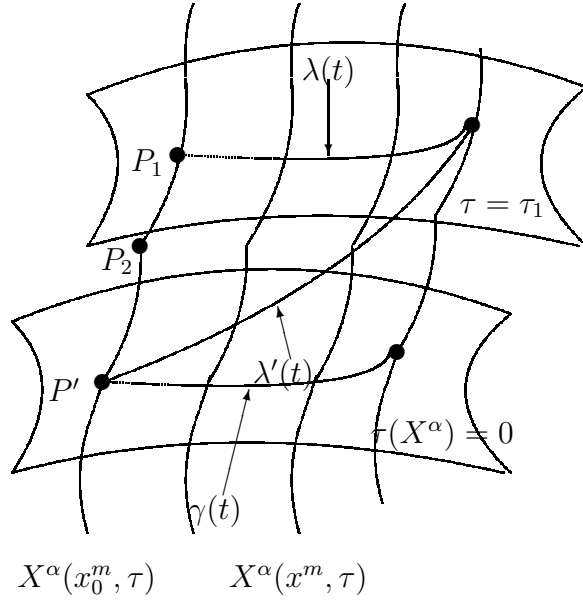


Figure 1: On the introduction of the exponential map.  $P' = \phi_{\tau_1}(P_1) = \phi_{\tau_2}(P_2)$ .  $P' \in \mathcal{M}_4$  ( $\mathcal{M}_4 : \tau(X^\alpha) = 0$ )

Now we have to discuss how the vectors and tensors are transformed by the map  $\phi_\tau$ .

Let  $\mathbf{V}$  be a tangent vector to the curve  $\lambda(t)$  at the point  $P_1$ , having the following form in local coordinates in a neighborhood of the point  $P_1$ :

$$X^\alpha(\lambda(t)) = X^\alpha(P_1) + tV^\alpha, \quad (48)$$

where  $V^\alpha$  are the components of the vector  $\mathbf{V}$  ( $\mathbf{V} = \frac{\partial}{\partial t}$ ) in the coordinate basis  $\mathbf{E}_\alpha$ , i.e.

$$\mathbf{V} = V^\alpha \frac{\partial}{\partial X^\alpha}. \quad (49)$$

Comparing the equation (48) with the following series expansion:

$$\begin{aligned} X^\alpha(\lambda(t)) &= X^\alpha(x^m(\lambda(t)), \tau(\lambda(t))) = \\ &= X^\alpha(x_0^m, \tau_1) + \left( X^\alpha_{,m} \frac{dx^m}{dt} + \frac{\partial X^\alpha}{\partial \tau} \frac{d\tau}{dt} \right) \Big|_{P_1} \cdot t \\ &+ O(t^2), \end{aligned} \quad (50)$$

we obtain

$$V^\alpha|_{P_1} = \left( X^\alpha_{,m} \frac{dx^m}{dt} + \xi^\alpha \frac{d\tau}{dt} \right) \Big|_{P_1} \quad \left( \xi^\alpha = \frac{dX^\alpha}{d\tau} \Big|_{x^m=\text{const}} \right). \quad (51)$$

The curve  $\lambda(t)$  can be projected by the exponential map  $\phi_{\tau(\lambda)}$  onto the hypersurface  $\tau = 0$ . The notation  $\phi_{\tau(\lambda)}$  should accentuate that each point of the curve  $\lambda(t)$  is mapped by the proper

exponential map  $\phi_\tau$  ( $\tau$  depends on  $t$ ). We denote the mapped curve obtained by this procedure by  $\gamma(t)$ :

$$\phi(\lambda(t)) = \gamma(t), \quad (52)$$

where  $\phi$  means  $\phi_{\tau(\lambda(t))}$ .

Further we shall consider only vector fields  $\mathbf{V}$  commuting with  $\frac{\partial}{\partial \tau}$ , i.e. the vector fields being projectors. In this case the maps of curves  $\lambda(t)$  and  $\lambda'(t)$  ( $\lambda'(0) = P'$ ) coincide:

$$\phi(\lambda(t)) = \phi(\lambda'(t)) = \gamma(t), \quad (53)$$

where  $\lambda'(t) = \phi_{\tau_1}(\lambda(t))$ . Therefore, without any further restriction we may consider only such curves whose initial points  $P_1$  ( $P_1 = \lambda(0)$ ) belong to the hypersurface  $\tau = 0$ , i.e.  $P_1 = P' \in \mathcal{M}_4$ .

For the mapped curve  $\gamma(t)$  following expansion is valid

$$\begin{aligned} X^\alpha(\gamma(t)) &= X^\alpha \left[ \left( x_0^m + \frac{dx^m}{dt} + O(t^2) \right), 0 \right] \\ &= X^\alpha(x_0^m, 0) + \left( X^\alpha_{,m} \frac{dx^m}{dt} \right) \Big|_P \cdot t + O(t^2). \end{aligned} \quad (54)$$

From the equation (51) follows

$$V^\alpha = \left( \frac{dx^m}{dt} e_m + \frac{d\tau}{dt} \xi \right) (X^\alpha), \quad (55)$$

where the vectors are defined by

$$e_m = \frac{\partial}{\partial x^m}, \quad \xi = \frac{\partial}{\partial \tau}. \quad (56)$$

The equation (55) implies that the following relation for the vector field  $\mathbf{V}$  is fulfilled:

$$\mathbf{V} = \frac{dx^m}{dt} \mathbf{e}_m + \frac{d\tau}{dt} \xi. \quad (57)$$

Thus at the point  $P_1$  ( $P_1 \in \mathcal{M}_4$ ;  $\gamma(0) = \lambda'(0) = P_1$ ) the 4-dimensional  $T_{P_1}$  and 5-dimensional  $T_{\phi(P_1)}$  vector spaces can be constructed as follows [14]:

$$\mathbf{V} \in T_{P_1}, \quad \phi_* \mathbf{V} \in T_{\phi(P_1)}. \quad (58)$$

Here we used the abbreviations (compare with [14]):

$$\phi_* \mathbf{V} \equiv \left( \frac{\partial}{\partial t} \right) \Big|_{\gamma|_{\phi(P_1)}} = \frac{dx^m}{dt} \Big|_{P_1} \mathbf{e}_m, \quad (59)$$

where

$$\frac{dx^m}{dt} = x^m_{,\alpha} V^\alpha. \quad (60)$$

It is necessary to note that  $\phi(P_1) = P_1$ ,  $\gamma(0) = \lambda'(0) = P_1$ .

It is easy to show that the one-form  $\mathbf{e}^m \equiv dx^m$  and the vectors  $\mathbf{e}_m = \frac{\partial}{\partial x^m}$  satisfy the equations

$$\langle \boldsymbol{\zeta}, \mathbf{e}_m \rangle = 0, \quad \langle \mathbf{e}^m, \boldsymbol{\xi} \rangle = 0. \quad (61)$$

These equations imply that one can rewrite the projector  $h^\alpha_\varepsilon$  in the form

$$h^\alpha_\varepsilon = g^\alpha_m g^m_\varepsilon = g^\alpha_\varepsilon - \xi^\alpha \zeta_\varepsilon, \quad (62)$$

where we used the definitions [6]

$$\text{a) } g^m_\varepsilon = \langle \mathbf{e}^m, \mathbf{e}_\varepsilon \rangle = x^m_{,\varepsilon}, \quad \text{b) } g^\varepsilon_m = \langle \mathbf{e}^\varepsilon, \mathbf{e}_m \rangle = X^\varepsilon_{,m}. \quad (63)$$

Apart from that it is easy to show that between the quantities  $\mathbf{e}_m$ ,  $\mathbf{e}^m$ ,  $\mathbf{e}_\varepsilon$  and  $\mathbf{e}^\varepsilon$  the following relation is valid:

$$\text{a) } \mathbf{e}_\varepsilon = g^m_\varepsilon \mathbf{e}_m + \zeta_\varepsilon \boldsymbol{\xi}, \quad \text{b) } \mathbf{e}^\varepsilon = g^\varepsilon_m dx^m + \xi^\varepsilon d\tau. \quad (64)$$

The last relation and the definition (59) lead us to

$$\tilde{\mathbf{V}} \equiv \phi_* \mathbf{V} = (g^m_\alpha V^\alpha) \mathbf{e}_m = \tilde{V}^\varepsilon \mathbf{e}_\varepsilon, \quad (65)$$

where we used the abbreviation

$$\tilde{V}^\varepsilon \equiv h^\varepsilon_\alpha V^\alpha. \quad (66)$$

Thus in the tangent vector space  $T_P$  it is possible to define a 4-dimensional subspace  $\tilde{T}_{P_1}$  ( $\tilde{T}_{P_1} \subset T_{P_1}$ ):

$$\tilde{T}_P = \{ \tilde{\mathbf{V}} : \forall \tilde{\mathbf{V}} \in \tilde{T}_P, \quad \phi_* \tilde{\mathbf{V}} = \tilde{\mathbf{V}} \}. \quad (67)$$

The equation  $\phi_* \mathbf{V} = \tilde{V}^\varepsilon \mathbf{e}_\varepsilon$  should be interpreted in the following way:

$$\phi_* \mathbf{V} = (x^m_{,\alpha} V^\alpha) \mathbf{e}_m \in T(\mathcal{M}_4). \quad (68)$$

In a vector space  $T(\mathcal{M}_5)$  it is possible to construct a 4-dimensional vector space formed by the vectors of the type  $(h^\varepsilon_\alpha V^\alpha) \mathbf{e}_\varepsilon$ . The spaces  $T(\mathcal{M}_4)$  and  $\tilde{T}(\mathcal{M}_5)$  are isomorphic:

$$(x^m_{,\alpha} V^\alpha) \mathbf{e}_m \iff (h^\varepsilon_\alpha V^\alpha) \mathbf{e}_\varepsilon. \quad (69)$$

The map  $\phi_*$ , namely

$$T_P(\mathcal{M}_5) \xrightarrow{\phi_*} T_{\phi(P)}(\mathcal{M}_4),$$

naturally induces the map  $\phi^*$  for the one-forms:

$$T^*_{\phi(P)}(\mathcal{M}_4) \xrightarrow{\phi^*} T^*_P(\mathcal{M}_5),$$

where for all  $\omega \in T_{\phi(P_1)}^*$  and for all  $V \in T_{P_1}$  the next relation is valid:

$$\langle \phi^* \omega, V \rangle|_{P_1} = \langle \omega, \phi_* V \rangle|_{\phi(P_1)}. \quad (70)$$

The set of all one-forms satisfying the relation

$$\phi^* \tilde{\omega} = \tilde{\omega} \quad (71)$$

forms a linear space  $\tilde{T}^*(\mathcal{M}_5) \subset T^*(\mathcal{M}_5)$ , where  $T^*(\mathcal{M}_5)$  is the space of all one-forms. From (70) follows that for any one-form from  $\tilde{T}^*(\mathcal{M}_5)$  holds

$$\langle \tilde{\omega}, \xi \rangle = 0, \quad (\tilde{\omega}_\alpha X^\alpha = 0). \quad (72)$$

There are two possible ways to associate elements of  $T^*(\mathcal{M}_5)$  with elements of  $\tilde{T}^*(\mathcal{M}_5)$ :

$$\text{a) } \omega_\alpha \longrightarrow \tilde{\omega}_\alpha = h^\varepsilon_\alpha \omega_\varepsilon,$$

$$\text{b) } \omega_\alpha \longrightarrow \tilde{\omega}_\alpha = P^\varepsilon_\alpha \omega_\varepsilon.$$

In both cases the quantities  $\tilde{\omega}_\alpha$  satisfy the equation (72) automatically. Hereinafter quantities with tilde will be associated with physical quantities in the space-time. However, the definition a) cannot be accepted, as in this case the following relations would be valid:

$$\tilde{V}^\alpha \equiv h^\alpha_\varepsilon V^\varepsilon \neq \tilde{g}^{\alpha\varepsilon} \tilde{M}_\varepsilon, \quad \tilde{M}_\alpha \equiv h^\varepsilon_\alpha V_\varepsilon \neq \tilde{g}_{\varepsilon\alpha} \tilde{V}^\varepsilon,$$

where

$$\tilde{g}_{\alpha\beta} = h^\varepsilon_\alpha h^\sigma_\beta g_{\varepsilon\sigma}, \quad \tilde{g}^{\alpha\sigma} = h^\alpha_\nu h^\sigma_\mu g^{\nu\mu}.$$

On the contrary, the definition b) is consistent. In this case the following relations will be valid:

$$\begin{aligned} \text{a) } \tilde{g}_{\alpha\beta} &\equiv P^\varepsilon_\alpha P^\sigma_\beta g_{\varepsilon\sigma} = P_{\alpha\beta}, \\ \text{b) } \tilde{g}^{\alpha\sigma} &\equiv h^\alpha_\nu h^\sigma_\mu g^{\nu\mu} = g^{\alpha\sigma} - 2X^{(\alpha} \zeta^{\sigma)} + \frac{1}{\Lambda^2} X^\alpha X^\sigma, \\ \text{c) } \tilde{g}^\mu_\nu &\equiv h^\mu_\varepsilon P^\alpha_\nu g^\varepsilon_\alpha = h^\mu_\nu. \end{aligned} \quad (73)$$

Using (73), for an arbitrary vector  $V$  we obtain:

$$\tilde{V}^\alpha = \tilde{g}^{\alpha\varepsilon} \tilde{V}_\varepsilon, \quad \tilde{V}_\alpha = \tilde{g}_{\alpha\varepsilon} \tilde{V}^\varepsilon. \quad (74)$$

The last results can be summarized in the sentence: The 5-dimensional tensors are to be projected onto hypersurface  $\tau(X^\alpha) = 0$  (projected quantities are denoted by a tilde) with the help of the procedure:

$$T^{\mu\cdots\cdots\nu} \xrightarrow{\phi} \tilde{T}^{\mu\cdots\cdots\nu} \equiv h^\mu_\sigma \cdots P^\tau_\nu \cdots T^{\sigma\cdots\cdots\tau}. \quad (75)$$

The quantities  $x^i$  being introduced as parameters earlier and parametrizing the congruence (23) can be used as coordinates in  $\mathcal{M}_4$ . Let us point out that it is necessary to require a certain

continuity for the quantities  $x^i$ . Apart from that these quantities are defined accurately within the following transformation:  $x^i \longrightarrow x^{i'} = x^{i'}(x^j)$ . In this case the vectors  $\mathbf{e}_m = \frac{\partial}{\partial x^m}$  and the one-forms  $\mathbf{e}^m = dx^m$  form a basis in  $T(\mathcal{M}_4)$  and  $T^*(\mathcal{M}_4)$ , either. These bases satisfy the following relations:

$$\langle \mathbf{e}^m, \mathbf{e}_n \rangle = \delta^m_n, \quad [\mathcal{R}, \mathbf{e}_m] = 0, \quad \langle \boldsymbol{\nu}, \mathbf{e}_m \rangle = 0. \quad (76)$$

Using the equations (44), (63) and (64), we can find several important relations:

$$\text{a) } \mathbf{e}_\alpha h^\alpha_\varepsilon = g^m_\varepsilon \mathbf{e}_m, \quad \text{b) } \mathbf{e}^\alpha h^\varepsilon_\alpha = g^\varepsilon_m \mathbf{e}^m; \quad (77)$$

$$\text{a) } \mathbf{e}_\alpha P^\alpha_\varepsilon = g^m_\varepsilon \mathbf{e}_m + (\zeta_\varepsilon - s_\varepsilon) \mathbf{s}, \quad \text{b) } \mathbf{e}^\alpha P^\varepsilon_\alpha = (g^\alpha_m P^\varepsilon_\alpha) \mathbf{e}^m; \quad (78)$$

$$\text{a) } \mathbf{e}_\alpha b^\alpha_\varepsilon = (g^m_\alpha b^\alpha_\varepsilon) \mathbf{e}_m, \quad \text{b) } \mathbf{e}^\alpha b^\varepsilon_\alpha = g^\varepsilon_m \mathbf{e}^m + \mathcal{X}^\varepsilon d\tau. \quad (79)$$

We already mentioned that the tangent spaces  $T(\mathcal{M}_4)$  and  $\tilde{T}(\mathcal{M}_5)$  are isomorphic. Therefore one can write:

$$T^m \equiv g^m_\varepsilon \tilde{T}^\varepsilon, \quad \tilde{T}^\varepsilon = g^\varepsilon_m T^m, \quad (80)$$

$$\omega_m \equiv g^\varepsilon_m \tilde{\omega}, \quad \tilde{\omega}_\varepsilon = g^m_\varepsilon \omega_m. \quad (81)$$

Thus the projection procedure from  $T(\mathcal{M}_5)$  into  $T(\mathcal{M}_4)$  is defined as follows:

$$T^\alpha \longrightarrow T^m = \tilde{g}^m_\varepsilon T^\varepsilon, \quad \omega_\alpha \longrightarrow \omega_m = \tilde{g}^\varepsilon_m \omega_\varepsilon, \quad (82)$$

where we used the abbreviation

$$\tilde{g}^\varepsilon_m \equiv P^\varepsilon_\alpha g^\alpha_m \neq g^\varepsilon_m, \quad \tilde{g}^m_\varepsilon \equiv h^\alpha_\varepsilon g^m_\alpha = g^m_\varepsilon. \quad (83)$$

The metric induced on the hypersurface  $\mathcal{M}_4$  will be denoted further by  $\tilde{\mathbf{g}}$  (in the theory of surfaces one understands under the induced metric the quantity  $b_{\mu\nu}$  defined by means of (41a) [14]). This metric satisfies the following relations:

$$\begin{aligned} \text{a) } \tilde{g}_{mn} &\equiv \tilde{g}^\alpha_m \tilde{g}^\beta_n g_{\alpha\beta}, & \text{b) } \tilde{g}^{mn} &\equiv \tilde{g}^m_\alpha \tilde{g}^n_\beta g^{\alpha\beta}, \\ \text{c) } \tilde{g}^m_n &\equiv \tilde{g}^m_\alpha \tilde{g}^\varepsilon_n g^\alpha_\varepsilon = \delta^m_n, & \text{d) } \tilde{g}^{mn} \tilde{g}_{mk} &= \delta^n_k. \end{aligned} \quad (84)$$

It is necessary to accentuate that the relation

$$\mathbf{g}(\mathbf{e}_m, \mathbf{e}_n) \neq \tilde{g}_{mn}, \quad \tilde{g}_{mn} = \tilde{\mathbf{g}}(\mathbf{e}_m, \mathbf{e}_n) \quad (85)$$

is fulfilled, where

$$\mathbf{g} \equiv g_{\alpha\varepsilon} dX^\alpha \otimes dX^\varepsilon, \quad \tilde{\mathbf{g}} \equiv \tilde{g}_{mn} dx^m \otimes dx^n. \quad (86)$$

At the end we have to present further two important relations that immediately follow from (76):

$$\text{a) } x^m_{,\alpha} X^\alpha = 0, \quad \text{b) } X^\varepsilon_{,m} \zeta_\varepsilon = 0. \quad (87)$$

### 3 Field Equations

In the introduction we mentioned that in the course of four decades three neighboring versions of Schmutzer's 5-dimensional Projective Unified Field Theory came into being. All these versions are based on the following 5-dimensional field equations:

$$\mathcal{G}_{\alpha\varepsilon} = \kappa_0 \theta_{\alpha\varepsilon}. \quad (88)$$

The explicit expression of the symmetric tensor  $\mathcal{G}_{\alpha\varepsilon}$  for the versions II and III of PUFT can be found, using the equations (1) and (13), respectively. In order to obtain a 4-dimensional field equations these 5-dimensional equations have to be projected onto the 4-dimensional space-time. The equation (88) can always be written in the following form:

$$\tilde{\mathcal{G}}_{\mu\nu} + 2 \tilde{\mathcal{G}}_{(\mu} s_{\nu)} + \mathcal{G} s_{\mu} s_{\nu} = \kappa_0 (\tilde{\overset{5}{\theta}}_{\mu\nu} + 2 \tilde{\overset{5}{\theta}}_{(\mu} s_{\nu)} + \tilde{\overset{5}{\theta}} s_{\mu} s_{\nu}), \quad (89)$$

where the abbreviations are given by

$$\begin{aligned} \text{a) } \tilde{\mathcal{G}}_{\mu\nu} &\equiv P^\alpha_\mu P^\beta_\nu \mathcal{G}_{\alpha\beta}, & \text{b) } \tilde{\mathcal{G}}_\mu &\equiv P^\alpha_\mu s^\beta \mathcal{G}_{\alpha\beta}, & \text{c) } \mathcal{G} &\equiv s^\alpha s^\beta \mathcal{G}_{\alpha\beta}, \\ \text{d) } \tilde{\overset{5}{\theta}}_{\mu\nu} &\equiv P^\alpha_\mu P^\beta_\nu \theta_{\alpha\beta}, & \text{e) } \tilde{\overset{5}{\theta}}_\mu &\equiv P^\alpha_\mu s^\beta \theta_{\alpha\beta}, & \text{f) } \tilde{\overset{5}{\theta}} &\equiv s^\alpha s^\beta \theta_{\alpha\beta}. \end{aligned} \quad (90)$$

It is easy to see that the equation (89) is equivalent to the following set of equations:

$$\begin{aligned} \text{a) } \tilde{\mathcal{G}}_{\mu\nu} &= \kappa_0 \tilde{\overset{5}{\theta}}_{\mu\nu}, \\ \text{b) } \tilde{\mathcal{G}}_\mu &= \kappa_0 \tilde{\overset{5}{\theta}}_\mu, \\ \text{c) } \mathcal{G} &= \kappa_0 \tilde{\overset{5}{\theta}} \end{aligned} \quad (91)$$

Further we can see that the following correspondence is valid:

$$\text{equation (91a)} \iff \text{generalized Einstein equations,}$$

$$\text{equation (91b)} \iff \text{generalized Maxwell equations,}$$

$$\text{equation (91c)} \iff \text{Field equation of the scalaric field.}$$

Henceforth the 4-dimensional hypersurface  $\mathcal{M}_4$  will be identified with the space-time. The physical metrics  $\overset{4}{\mathbf{g}}$  of the space-time can be defined in different ways (all space-time quantities will be denoted by an index "4"). For example, we can identify the 4-dimensional physical metric  $\overset{4}{\mathbf{g}}$  with the metric  $\tilde{\mathbf{g}}$  induced on the hypersurface  $\mathcal{M}_4$ :

$$\overset{4}{\mathbf{g}} = \tilde{\mathbf{g}} \quad (\overset{4}{g}_{\mu\nu} = \tilde{g}_{\mu\nu}, \quad \overset{4}{g}^{\mu\nu} = \tilde{g}^{\mu\nu}). \quad (92)$$

In this case we obtain version I or III of PUFT if we use the 5-dimensional equations (1) or (13), respectively. However, it is physically possible to connect these metrics  $\overset{4}{g}$  and  $\tilde{g}$  by a conformal transformation:

$$\overset{4}{g} = e^\sigma \tilde{g} \quad (\overset{4}{g}_{\mu\nu} = e^\sigma \tilde{g}_{\mu\nu}, \quad \overset{4}{g}^{\mu\nu} = e^{-\sigma} \tilde{g}^{\mu\nu}). \quad (93)$$

In this case the 5-dimensional Einstein-like equations (1) lead to the system of equations of version II of PUFT. In order to consider both these cases simultaneously we rewrite the equations (92) and (93) in the form:

$$\overset{4}{g} = e^{\epsilon\sigma} \tilde{g} \quad \text{with} \quad \epsilon = \begin{cases} 0 & \longleftrightarrow \text{Version I+Version III} \\ 1 & \longleftrightarrow \text{Version II} \end{cases} \quad (94)$$

The projection formalism can be simplified by using a non-Riemannian connection in the 5-dimensional space and considering the Riemannian connection in the 4-dimensional space-time as induced.

### 3.1 Connection on $\mathcal{M}_5$

Let us introduce an induced (affine) connection on the hypersurface  $\mathcal{M}_4$  denoted hereinafter as  $\overset{4}{\nabla}$ . The induced connection  $\overset{4}{\nabla}$  and the connection on  $\mathcal{M}_5$  (denoted as  $\overset{5}{\nabla}$ ) are connected in the following way:

$$\overset{4}{\nabla}_\varepsilon \tilde{T}^{\mu\dots}_{\nu\dots} = h^\mu_\sigma \dots P^\alpha_\varepsilon P^\lambda_\nu \dots \tilde{T}^{\sigma\dots}_{\lambda\dots||\alpha}, \quad (95)$$

where

$$T^{\dots}_{\dots||\alpha} = \overset{5}{\nabla}_\alpha T^{\dots}, \quad \overset{5}{\nabla}_\alpha \equiv \overset{5}{\nabla} e_\alpha, \quad \overset{4}{\nabla}_\alpha \equiv \overset{4}{\nabla} e_\alpha. \quad (96)$$

Henceforth we assume that the connection on  $\mathcal{M}_4$  is Riemannian, i.e. metrical and symmetric:

$$\text{a) } \overset{4}{\nabla}_\varepsilon \overset{4}{g}_{\mu\nu} = 0, \quad \text{b) } \overset{4}{\nabla}_\alpha \overset{4}{\nabla}_\varepsilon f = \overset{4}{\nabla}_\varepsilon \overset{4}{\nabla}_\alpha f. \quad (97)$$

Since the 4-dimensional covariant derivative is defined only for the projected vectors (see (95)), the function  $f$  should satisfy the condition (26):  $\mathcal{L}_{\mathcal{R}} f = f_{,\alpha} X^\alpha = 0$ . The 4-dimensional covariant derivative (with respect to  $\overset{4}{\nabla}$ ) in the direction of the basis vectors  $e_m$  ( $e_m = \frac{\partial}{\partial x^m}$ ) will be denoted by a semicolon:

$$\overset{4}{\nabla}_{e_m} \tilde{T}^k \equiv \overset{4}{\nabla}_m \tilde{T}^k \equiv \tilde{T}^k_{;m} = g^k_\alpha g^\varepsilon_m \overset{4}{\nabla}_\varepsilon \tilde{T}^\alpha, \quad (98)$$

where  $\tilde{T} = e_m \tilde{T}^m = e_\alpha \tilde{T}^\alpha$ .

As it is well known, the Riemannian connection is completely defined by means of a metric. Therefore the relations (97) are in fact conditions for the 5-dimensional connection  $\overset{5}{\nabla}$ . In

particular, from (97) follows that the 5-dimensional connection  $\overset{5}{\nabla}$  has to satisfy the following relations:

$$\begin{aligned} \text{a)} \quad & \tilde{Q}_{\varepsilon\alpha\beta} \equiv P^\tau_\varepsilon P^\nu_\alpha P^\mu_\beta Q_{\tau\nu\mu} = \epsilon\sigma_{,\varepsilon} \tilde{g}_{\alpha\beta} = \epsilon\sigma_{,\varepsilon} P_{\alpha\beta}, \\ \text{b)} \quad & \tilde{S}_{\alpha\beta}{}^\tau \equiv P^\nu_\alpha P^\mu_\beta h^\tau_\varepsilon S_{\nu\mu}{}^\varepsilon = 0, \\ \text{c)} \quad & h^\mu_\alpha P^\nu_\beta X^\alpha_{||\nu} = 0, \end{aligned} \tag{99}$$

where the usual definitions

$$\text{a)} \quad g_{\alpha\beta||\varepsilon} = -Q_{\varepsilon\alpha\beta}, \quad \text{b)} \quad g^{\alpha\beta}{}_{||\varepsilon} = Q_\varepsilon^{\alpha\beta} \quad \text{c)} \quad S_{\alpha\beta}{}^\gamma = \Gamma_{[\alpha\beta]}{}^\gamma \tag{100}$$

are used. One can easily verify that the 5-dimensional connection in general is nonsymmetric and nonmetrival.

For this reason we write the 5-dimensional connection coefficients  $\Gamma^\varepsilon_{\mu\nu}$  in the following form:

$$\Gamma^\varepsilon_{\mu\nu} = \left\{ \begin{matrix} \varepsilon \\ \mu\nu \end{matrix} \right\} + \sigma_{\mu\nu}{}^\varepsilon, \tag{101}$$

where

$$\sigma_{\mu\nu}{}^\varepsilon = -(S^\varepsilon_{\nu\mu} + S_{\nu\mu}{}^\varepsilon - S_{\mu\nu}{}^\varepsilon) + \frac{1}{2}(Q_{\nu\mu}{}^\varepsilon + Q_{\mu\nu}{}^\varepsilon - Q^\varepsilon_{\nu\mu}). \tag{102}$$

The 5-dimensional connection cannot be found uniquely from the demands (99). However, the 5-dimensional field equations (see (1) and (13)) only contain Riemannian covariant derivatives, and therefore, the 5-dimensional connection  $\overset{5}{\nabla}$  is only an auxiliary quantity. Thus within some restrictions, the 5-dimensional connection coefficients  $\Gamma^\varepsilon_{\mu\nu}$  can be chosen arbitrarily. Therefore we choose the 5-dimensional connection on  $\mathcal{M}_5$  in a certain way to make calculations as simple as possible. First let us in general assume:

$$\overset{5}{\nabla}_\xi \mathbf{e}_\varepsilon = -\Upsilon_\varepsilon{}^\nu \mathbf{e}_\nu \quad (\Gamma^\varepsilon_{\mu\nu} X^\nu = -\Upsilon_\mu{}^\varepsilon), \tag{103}$$

where  $\Upsilon_\mu{}^\varepsilon$  is an arbitrary projector. Taking into account the relation

$$\left\{ \begin{matrix} \varepsilon \\ \mu\nu \end{matrix} \right\} X^\mu = -g^\varepsilon_\nu + \frac{1}{2} X^\varepsilon_\nu, \tag{104}$$

which follows immediately from (28), we obtain

$$\sigma_{\mu\nu}{}^\varepsilon X^\nu = -\Sigma_\mu{}^\varepsilon - \frac{1}{2} X^\varepsilon_\mu, \tag{105}$$

where we introduced the abbreviation

$$\Sigma_\mu{}^\varepsilon \equiv \Upsilon_\mu{}^\varepsilon - g_\mu{}^\varepsilon. \tag{106}$$



From the conditions (99b) and (99c) we obtain the following relation for the torsion tensor:

$$S_{\alpha\beta}{}^\gamma = A_{\alpha\beta}X^\gamma + \frac{1}{2S^2}h^\gamma{}_\tau(X_\alpha\Sigma_\beta{}^\tau - X_\beta\Sigma_\alpha{}^\tau). \quad (107)$$

Here we used the abbreviation

$$A_{\varepsilon\tau} = S_{\varepsilon\tau}{}^\mu\zeta_\mu. \quad (108)$$

It is possible to show that the 5-dimensional connection has the simplest form if according to (99a) we put

$$Q_{\varepsilon\alpha\beta} = \epsilon\sigma_{,\varepsilon}P_{\alpha\beta}. \quad (109)$$

In this case the quantity  $\Sigma_{\mu\nu}$  will be antisymmetric (with no other limitations):

$$\Sigma_{\mu\nu} = -\Sigma_{\nu\mu}. \quad (110)$$

The tensors  $S_{\alpha\beta}{}^\gamma$  and  $\sigma_{\alpha\beta}{}^\gamma$  in this case are given by:

$$S_{\alpha\beta}{}^\gamma = -\frac{X^\gamma}{2S^2}[X_{\alpha\beta} + (s_\alpha S_{,\beta} - s_\beta S_{,\alpha})] + \frac{1}{2S^2}(X_\alpha\Sigma_\beta{}^\gamma - X_\beta\Sigma_\alpha{}^\gamma), \quad (111)$$

$$\begin{aligned} \sigma_{\lambda\mu\varepsilon} = & -\frac{1}{2S}[(X_{\lambda\mu}s_\varepsilon - X_{\varepsilon\lambda}s_\mu + X_{\mu\varepsilon}s_\lambda) \\ & + 2s_\mu(s_\lambda S_{,\varepsilon} - s_\varepsilon S_{,\lambda})] - \frac{s_\mu}{S}\Sigma_{\lambda\varepsilon} \\ & - \frac{\epsilon}{2S}(P_{\lambda\mu}S_{,\varepsilon} - P_{\mu\varepsilon}S_{,\lambda} - P_{\lambda\varepsilon}S_{,\mu}). \end{aligned} \quad (112)$$

From the last relation follows that the connection on  $\mathcal{M}_5$  has the simplest form if the quantity  $\Sigma_{\mu\nu}$  is defined according to (110) as follows:

$$\Sigma_{\varepsilon\nu} = G_\varepsilon X_\nu - G_\nu X_\varepsilon, \quad (113)$$

where the abbreviation (33a) was used. Substituting the last expression into the relations (111) and (112), we obtain:

$$S_{\alpha\beta}{}^\sigma = s^\sigma\omega_{\alpha\beta}, \quad (114)$$

$$\begin{aligned} \sigma_{\varepsilon\tau\nu} = & \omega_{\varepsilon\tau}s_\nu - \omega_{\nu\varepsilon}s_\tau + \omega_{\tau\nu}s_\varepsilon \\ & - \frac{\epsilon}{2}(G_\varepsilon P_{\tau\nu} + G_\tau P_{\nu\varepsilon} - G_\nu P_{\tau\varepsilon}). \end{aligned} \quad (115)$$

At the end of this section we would like to point out once again that the connection on  $\mathcal{M}_5$  is an intermediate quantity. Its choice does not lead to any physical consequences. It can be shown that for any choice of  $\Sigma_{\varepsilon\alpha}$  and  $Q_\nu{}^{\varepsilon\alpha}$  (these quantities have to satisfy the conditions (99) only) the 4-dimensional physical equations get the same form. However, in the general case all calculations become unwieldy. Therefore we don't present them here fully; hereinafter we only will consider the case (109) and (113). Thus the torsion tensor  $S_{\alpha\beta}{}^\varepsilon$  and the tensor  $\sigma_{\alpha\beta}{}^\varepsilon$  take the simplest form, i.e. (114) and (115), respectively.

### 3.2 Projection of the Curvature Tensor and Related Quantities

Now we have to analyse the equation (91). In order to do it, we can use the general relation

$$\tilde{T}_{\varepsilon||\mu||\lambda} - \tilde{T}_{\varepsilon||\lambda||\mu} = \tilde{T}_\nu G^\nu_{\varepsilon\mu\lambda} + 2 \tilde{T}_{\varepsilon||\alpha} S_{\lambda\mu}^\alpha, \quad (116)$$

where

$$G^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\alpha_{\varrho\gamma} \Gamma^\varrho_{\beta\delta} - \Gamma^\alpha_{\varrho\delta} \Gamma^\varrho_{\beta\gamma}. \quad (117)$$

To project the equation (116) onto space-time we need the following two relations:

$$\begin{aligned} \text{a)} \quad \nabla_\beta^4 \nabla_\alpha^4 \tilde{T}_\sigma &= \tilde{T}_{\delta||\varepsilon||\nu} P^\nu_\beta P^\delta_\sigma P^\varepsilon_\alpha, \\ \text{b)} \quad \nabla_\beta^4 \nabla_\alpha^4 \tilde{T}^\sigma &= \tilde{T}^\delta_{||\varepsilon||\nu} P^\nu_\beta P^\varepsilon_\alpha h^\sigma_\delta. \end{aligned} \quad (118)$$

The relation (118a) follows immediately from the equation

$$\nabla_\beta^4 \nabla_\alpha^4 \tilde{T}_\sigma = (\tilde{T}_{\gamma||\tau} P^\gamma_\delta P^\tau_\varepsilon)_{||\nu} P^\nu_\beta P^\delta_\sigma P^\varepsilon_\alpha, \quad (119)$$

in which the covariant derivatives

$$\text{a)} \quad s^\mu_{||\nu} = G^\mu s_\nu, \quad \text{b)} \quad s_{\mu||\nu} = G_\mu s_\nu \quad (120)$$

are substituted. The equation (118b) can be similarly proved. Proceeding further, we suppose that the vector  $\tilde{T}_\varepsilon$  satisfies the projector condition (27). The equation

$$s^\delta \tilde{T}_{\varepsilon||\delta} = -(G^\lambda \tilde{T}_\lambda) s_\varepsilon. \quad (121)$$

is fulfilled in this case. Using the relations (114), (120) and (121) we obtain the interesting equality

$$\nabla_\beta^4 \nabla_\alpha^4 \tilde{T}_\sigma - \nabla_\alpha^4 \nabla_\beta^4 \tilde{T}_\sigma = \tilde{G}^\nu_{\sigma\alpha\beta} \tilde{T}_\nu, \quad (122)$$

where according to (75) we used the abbreviation

$$\tilde{G}^\nu_{\sigma\alpha\beta} \equiv h^\nu_\mu P^\gamma_\sigma P^\delta_\alpha P^\varrho_\beta G^\mu_{\gamma\delta\varrho}. \quad (123)$$

The equation (122) being written in the basis  $\mathbf{e}_n$  (see (98)) is given by:

$$T_{s;a;b} - T_{s;b;a} = T_n G^n_{sab}, \quad (124)$$

where

$$T_n = g^\alpha_n \tilde{T}_\alpha, \quad G^n_{sab} = g^\nu_n g^\sigma_s g^\alpha_a g^\beta_b \tilde{G}^\nu_{\sigma\alpha\beta}. \quad (125)$$

As the equation (124) is correct for all space-time vectors, the following relation is valid:

$${}^4\tilde{R}^n_{sab} = G^n_{sab}. \quad (126)$$

Here  ${}^4\tilde{R}^n_{sab}$  is the 4-dimensional Riemannian curvature tensor

$${}^4\tilde{R}^a_{mnk} \equiv \left\{ \begin{matrix} a \\ mk \end{matrix} \right\}_{,n} - \left\{ \begin{matrix} a \\ mn \end{matrix} \right\}_{,k} + \left\{ \begin{matrix} t \\ mk \end{matrix} \right\} \left\{ \begin{matrix} a \\ tn \end{matrix} \right\} - \left\{ \begin{matrix} t \\ mn \end{matrix} \right\} \left\{ \begin{matrix} a \\ tk \end{matrix} \right\}, \quad (127)$$

where we used the usual definition

$$\left\{ \begin{matrix} k \\ at \end{matrix} \right\} \equiv \frac{1}{2} {}^4g^{ks} ({}^4g_{sa,t} + {}^4g_{ts,a} - {}^4g_{at,s}). \quad (128)$$

Obtaining the equation (124) we applied the following relation:

$$\nabla_\nu \tilde{T}_\mu g^\mu_m g^\nu_n = T_{m;n} \quad (T_m = g^\nu_m \tilde{T}_\nu). \quad (129)$$

Let  $T_\mu$  be an arbitrary one-form (covariant vector). According to the projection formalism developed above we can project this one-form onto the hypersurface  $\mathcal{M}_4$ :  $T^\mu \longrightarrow \tilde{T}_\mu \equiv P^\alpha_\mu T_\alpha$ . Then the equations (101) and (118) imply:

$$\nabla_\lambda \nabla_\varepsilon \tilde{T}_\mu = P^\alpha_\mu P^\beta_\varepsilon P^\gamma_\lambda (\tilde{T}_{\alpha;\beta} - \sigma_{\alpha\beta}{}^\nu \tilde{T}_\nu)_{||\gamma}. \quad (130)$$

Substituting the relations (33), (35), (115), (120), (121) as well as

$$a) P_{\mu\nu||\varepsilon} = \epsilon G_\varepsilon P_{\mu\nu} - s_\varepsilon (G_\mu s_\nu + G_\nu s_\mu), \quad b) P^\nu_{\mu||\varepsilon} = -s_\varepsilon (G^\nu s_\mu + G_\mu s^\nu), \quad (131)$$

$$s^\delta \tilde{T}_{\varepsilon;\delta} = s^\delta \tilde{T}_{\delta;\varepsilon} = (\omega_\varepsilon^\lambda - s_\varepsilon G^\lambda) \tilde{T}_\lambda, \quad (132)$$

$$G^\varrho_{||\varepsilon} = -\epsilon G_\varepsilon G^\varrho + g^{\varrho\lambda} G_{\lambda||\varepsilon} \quad (133)$$

into the last formula, we obtain the result

$$\begin{aligned} \nabla_\lambda \nabla_\varepsilon \tilde{T}_\mu - \nabla_\varepsilon \nabla_\lambda \tilde{T}_\mu &= \tilde{T}_\varrho \left\{ \tilde{R}^{\varrho}_{\mu\varepsilon\lambda} - 2\omega_{\varepsilon\lambda}\omega_\mu^\varrho + \omega_{\mu\varepsilon}\omega_\lambda^\varrho - \omega_{\mu\lambda}\omega_\varepsilon^\varrho \right. \\ &\quad - \frac{\epsilon}{2} \left[ -(\nabla_\lambda \tilde{G}_\mu) P^\varrho_\varepsilon + P_{\mu\varepsilon} g^{\varrho\alpha} \tilde{G}_{\alpha||\gamma} P^\gamma_\lambda + (\nabla_\varepsilon \tilde{G}_\mu) P^\varrho_\lambda \right. \\ &\quad - P_{\mu\lambda} g^{\varrho\alpha} \tilde{G}_{\alpha||\gamma} P^\gamma_\varepsilon \left. \right] + \frac{\epsilon^2}{4} \left[ -G^\varrho (G_\varepsilon P_{\mu\lambda} - G_\lambda P_{\mu\varepsilon}) \right. \\ &\quad + G_\varepsilon G_\mu P^\varrho_\lambda - G_\lambda G_\mu P^\varrho_\varepsilon \\ &\quad \left. - G^\nu G_\nu (P_{\mu\varepsilon} P^\varrho_\lambda - P_{\mu\lambda} P^\varrho_\varepsilon) \right] \left. \right\}. \end{aligned} \quad (134)$$

Now we are able to analyse the equation (91a). However, before doing it, let us summarize some formulas which are related to the projection formalism.

In the 5-dimensional space  $\mathcal{M}_5$  the basis vectors and basis one-forms were denoted by  $e_\mu$  and  $e^\mu$ , respectively. The 4-dimensional holonomic hypersurface  $\mathcal{M}_4$  in the 5-dimensional space  $\mathcal{M}_5$  is identified with the 4-dimensional space-time. The quantities projected onto the hypersurface  $\mathcal{M}_4$  were denoted by a tilde (see (75)). The quantities  $x^i$  parametrizing curves of the congruence (23) can be used as coordinates in the space-time. The tangent vectors  $e_i$  to the coordinate lines ( $e_i = \frac{\partial}{\partial x^i}$ ) of this 4-dimensional coordinate system form a 4-dimensional vector space  $\tilde{T}_P$  (see (67)). Between the basis vectors  $e_i$  and  $e_\mu$  exists consistency (64). Similar relations are valid for the dual basis  $e^i$  ( $e^i = dx^i$ ), too. Thus the 4-dimensional vectors and one-forms can be rewritten in the following form:

$$\begin{aligned} a) \quad \tilde{\mathbf{V}} &= \tilde{V}^\alpha e_\alpha = V^i e_i & (\tilde{V}^\alpha &= g^\alpha_i V^i, \quad V^i = g^i_\alpha \tilde{V}^\alpha) \\ b) \quad \tilde{\omega} &= \tilde{\omega}_\alpha e^\alpha = \omega_i e^i & (\tilde{\omega}_\alpha &= g^i_\alpha \omega_i, \quad \omega_i = g^\alpha_i \tilde{\omega}_\alpha). \end{aligned} \quad (135)$$

Let us remember that on the hypersurface  $\mathcal{M}_4$  we introduced two metrics: the induced metric  $\tilde{\mathbf{g}}$  and the physical metric  $\mathbf{g}$ . These metrics are connected by means of the relation (94). As the physical metric differs in general from the induced one, one should be careful in defining 4-dimensional physical quantities.

Using the abbreviation

$$\overset{4}{\omega}_{mn} \equiv \omega_{mn} = g^\mu_m g^\nu_n \tilde{\omega}_{\mu\nu} = g^\mu_m g^\nu_n \omega_{\mu\nu} \quad (136)$$

we obtain from (135) and (94) the following relations

$$a) \quad \omega_m{}^n \equiv g^\mu_n g^\nu_\mu \tilde{\omega}_\mu{}^\nu = e^{\epsilon\sigma} \overset{4}{\omega}_m{}^n, \quad b) \quad \omega^{mn} \equiv g^m_\mu g^n_\nu \tilde{\omega}^{\mu\nu} = e^{2\epsilon\sigma} \overset{4}{\omega}^{mn}, \quad (137)$$

where  $\overset{4}{\omega}_m{}^n = \overset{4}{g}^{nk} \overset{4}{\omega}_{mk}$  and  $\overset{4}{\omega}^{mn} = \overset{4}{g}^{mk} \overset{4}{g}^{nl} \overset{4}{\omega}_{kl}$ . In a similar way we deduce from (74) and (33a) the equations

$$a) \quad g^\mu_m G_\mu = -g^\mu_m \sigma_{,\mu} = -\sigma_{,m}, \quad b) \quad g^m_\mu \tilde{G}^\mu = -\tilde{g}^{\mu\nu} g^m_\nu \sigma_{,\mu} = -e^{\epsilon\sigma} \sigma^{,m}, \quad (138)$$

where  $\sigma^{,m} \equiv \overset{4}{g}^{mn} \sigma_{,n}$ .

Let us note that the space-time indices are to be moved with the help of the space-time metric  $\mathbf{g}$ .

It can be shown that for the arbitrarily projected quantities  $\tilde{\omega}$  and  $\tilde{\mathbf{V}}$  the relation

$$\tilde{\omega}_\tau \tilde{V}^\tau = \omega_n V^n \quad (\omega_m \equiv g^\mu_m \tilde{\omega}_\mu, \quad V^m \equiv g^m_\tau \tilde{V}^\tau) \quad (139)$$

is true. From (122) and (126) follows

$$g^\sigma_s g^\alpha_a g^\beta_b (\overset{4}{\nabla}_\beta \overset{4}{\nabla}_\alpha \tilde{T}_\sigma - \overset{4}{\nabla}_\alpha \overset{4}{\nabla}_\beta \tilde{T}_\sigma) = \overset{4}{R}^n_{sab} \overset{4}{T}_n \quad (\overset{4}{T}_n \equiv g^\sigma_n \tilde{T}_\sigma). \quad (140)$$

Using the relations (134) and further the relations (138) to (140), we obtain the final result

$$\begin{aligned}
\tilde{R}^a{}_{mkl} &\equiv g^a{}_\alpha g^\mu{}_m g^\varepsilon{}_k g^\lambda{}_l R^\alpha{}_{\mu\varepsilon\lambda} = \\
&= \overset{4}{R}^a{}_{mkl} + e^{\epsilon\sigma} \left( 2 \overset{4}{\omega}_{kl} \overset{4}{\omega}_m{}^a - \overset{4}{\omega}_{mk} \overset{4}{\omega}_l{}^a + \overset{4}{\omega}_{ml} \overset{4}{\omega}_k{}^a \right) \\
&+ \frac{\epsilon}{2} \left( \overset{4}{g}^{ak} \sigma_{,m;l} - \overset{4}{g}_{mk} \sigma_{,l}{}^a - \overset{4}{g}_l^a \sigma_{,m;k} + \overset{4}{g}_{ml} \sigma_{,k}{}^a \right) \\
&- \frac{\epsilon^2}{4} \left[ -\sigma_{,a} \left( \sigma_{,k} \overset{4}{g}_{ml} - \sigma_{,l} \overset{4}{g}_{mk} \right) + \overset{4}{g}_l^a \sigma_{,k} \sigma_{,m} \right. \\
&\left. - \overset{4}{g}_k^a \sigma_{,m} \sigma_{,l} - (\sigma_{,c} \sigma^{,c}) \left( \overset{4}{g}_{mk} \overset{4}{g}_l^a - \overset{4}{g}_{ml} \overset{4}{g}_k^a \right) \right].
\end{aligned} \tag{141}$$

In order to find the projection of the 5-dimensional Ricci tensor onto space-time  $\mathcal{M}_4$  let us consider the relation

$$\tilde{R}_{mn} \equiv g^\mu{}_m g^\nu{}_n \tilde{R}_{\mu\nu} = g^\mu{}_m g^\nu{}_n \left[ \tilde{R}^\varrho{}_{\mu\nu\varrho} + P^\alpha{}_\mu P^\beta{}_\nu (s_\varrho s^\sigma \tilde{R}^\varrho{}_{\alpha\beta\sigma}) \right], \tag{142}$$

where

$$\tilde{R}_{\mu\nu} \equiv P^\alpha{}_\mu P^\sigma{}_\nu \tilde{R}_{\alpha\sigma}, \quad \tilde{R}^\varrho{}_{\mu\nu\sigma} \equiv h^\varrho{}_\lambda P^\alpha{}_\mu P^\beta{}_\nu P^\tau{}_\sigma \tilde{R}^\lambda{}_{\mu\nu\sigma\tau}. \tag{143}$$

From (122), (126) and (134) we find

$$\begin{aligned}
g^\mu{}_m g^\nu{}_k \tilde{R}^\varrho{}_{\mu\nu\varrho} &= \overset{4}{R}_{mk} + 3e^{\epsilon\sigma} \overset{4}{\omega}_{ka} \overset{4}{\omega}_m{}^a - \frac{\epsilon}{2} \left( \sigma_{,m;k} + \frac{1}{2} \overset{4}{g}_{mk} \sigma_{,a}{}^a \right) \\
&+ \frac{\epsilon^2}{2} \left[ \overset{4}{g}_{mk} (\sigma_{,a} \sigma_{,a}) - \sigma_{,k} \sigma_{,m} \right].
\end{aligned} \tag{144}$$

The second term on the right hand side of the equation (142) can be calculated in the simplest way using the formulas (104) and (36). The result is

$$\tilde{R}^\sigma{}_{\alpha\beta\varrho} s_\sigma s^\varrho = \sigma_{,\beta;\alpha} + \sigma_{,\alpha} \sigma_{,\beta} - \omega_{\lambda\beta} \omega^\lambda{}_\alpha + \omega_{\lambda[\beta} s_{\alpha]} \sigma^{,\lambda} - s_\alpha s_\beta \sigma^{,\lambda} \sigma_{,\lambda}. \tag{145}$$

By substituting the expressions (144) and (145) into (142) we get the formula

$$\begin{aligned}
\tilde{R}_{mn} &= \overset{4}{R}_{mn} + 2e^{\epsilon\sigma} \overset{4}{\omega}_{na} \overset{4}{\omega}_m{}^a - \frac{\epsilon}{2} \overset{4}{g}_{mn} \sigma_{,a}{}^a + (1 + \epsilon - \frac{\epsilon^2}{2}) \sigma_{,m} \sigma_{,n} \\
&+ (1 - \epsilon) \left[ \sigma_{,m;n} - \frac{\epsilon}{2} \overset{4}{g}_{mn} \sigma_{,a}{}^a \right].
\end{aligned} \tag{146}$$

Here we used the equation

$$P^\alpha{}_\mu P^\beta{}_\nu \sigma_{,\beta;\alpha} = -\nabla_\mu \tilde{G}_\nu + \frac{\epsilon}{2} [2 \tilde{G}_\nu \tilde{G}_\mu - (G^\varrho \tilde{G}_\varrho) P_{\mu\nu}]. \tag{147}$$

By means of the expression

$$\tilde{R} = (P^{\mu\nu} + s^\mu s^\nu) \tilde{R}_{\mu\nu} \tag{148}$$

and taking into account the intermediate formulas

$$P^{\mu\nu} \stackrel{5}{R}_{\mu\nu} = e^{\epsilon\sigma} \stackrel{4}{g}{}^{mn} \stackrel{\sim}{R}_{mn}, \quad (149)$$

$$s^\mu s^\nu \stackrel{5}{R}_{\mu\nu} = \sigma^{\cdot\alpha}_{;\alpha} - \omega_{\lambda\alpha} \omega^{\lambda\alpha}, \quad (150)$$

$$\sigma^{\cdot\alpha}_{;\alpha} = e^{\epsilon\sigma} \left[ \sigma^{\cdot i}_{;i} + (1 - \epsilon) \sigma^{\cdot i} \sigma_{,i} \right], \quad (151)$$

we find the result

$$\stackrel{5}{R} = e^{\epsilon\sigma} \left( \stackrel{4}{R} + e^{\epsilon\sigma} \stackrel{4}{\omega}_{mn} \stackrel{4}{\omega}{}^{mn} - (2 - 3\epsilon) \sigma^{\cdot m}_{;m} + 2(1 - \epsilon + \frac{3}{4}\epsilon^2) \sigma^{\cdot m} \sigma_{,m} \right). \quad (152)$$

Now we immediately can obtain the 4-dimensional field equations of PUFT being restricted to the versions II and III of PUFT. Today the version I of PUFT has only historical value.

### 3.3 Version II

By projecting the 5-dimensional field equations (1) onto the 4-dimensional space-time with the help of the projection formalism developed above we obtain the 4-dimensional field equations of PUFT. As the space-time metric  $\stackrel{4}{g}$  is connected with the induced metric  $\tilde{g}$  on the hypersurface  $\mathcal{M}_4$  by means of (93), in case of the version II of PUFT it is necessary to put:

$$\left. \begin{aligned} G_{\mu\nu} &= \stackrel{5}{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \stackrel{5}{R} + \lambda_0 S_0 e^\sigma (g_{\mu\nu} + s_\mu s_\nu) \\ \epsilon &= 1 \end{aligned} \right\}. \quad (153)$$

#### 3.3.1 Generalized gravitational field equation

Using the last results obtained from the equation (91a) within the framework of (153), the field equations read:

$$\stackrel{4}{R}_{mn} - \frac{1}{2} \stackrel{4}{g}_{mn} \stackrel{4}{R} + \lambda_0 S_0 \stackrel{4}{g}_{mn} = \kappa_0 \stackrel{4}{T}_{mn}, \quad (154)$$

where

$$\begin{aligned} \stackrel{4}{T}_{mn} = \stackrel{4}{\theta}_{mn} + \frac{1}{\kappa_0} & \left[ 2e^\sigma (\stackrel{4}{\omega}_{ma} \stackrel{4}{\omega}{}^a_n + \frac{1}{4} \stackrel{4}{g}_{mn} \stackrel{4}{\omega}_{ab} \stackrel{4}{\omega}{}^{ab}) \right. \\ & \left. - \frac{3}{2} (\sigma_{,m} \sigma_{,n} - \frac{1}{2} \stackrel{4}{g}_{mn} \sigma^{\cdot a} \sigma_{,a}) \right] \end{aligned} \quad (155)$$

and

$$\stackrel{4}{\theta}_{mn} \equiv g^\mu{}_m g^\nu{}_n \stackrel{\sim}{\theta}_{\mu\nu}. \quad (156)$$

### 3.3.2 Generalized electromagnetic field equations (Maxwell equations)

Comparing the relation (155) with the expression (5), we find that the angular velocity of the congruence (23)  $\omega_{\alpha\sigma}$  is connected with the electromagnetic strength tensor in the following way:

$$B_{\alpha\sigma} \equiv \tilde{B}_{\alpha\sigma} = B_0 e^{a\sigma} \omega_{\alpha\sigma}, \quad (157)$$

where the constant  $B_0$  depends on the system of units. We choose the constant  $a$  in order to fulfill the next equation

$$B_{<\mu\nu||\alpha>} = 0. \quad (158)$$

It is easy to see that the relation

$$B_{<\mu\nu||\alpha>} = B_{<\mu\nu;\alpha>} \quad (159)$$

holds. Using the expression (36) and the equation

$$X_{<\mu\nu;\alpha>} = 0 \quad (160)$$

we find

$$B_{<\mu\nu;\alpha>} = B_0 e^{a\sigma} (1 + a) \omega_{<\alpha\mu\sigma,\nu>}. \quad (161)$$

This implies  $a = -1$  and

$$B_{\alpha\sigma} = B_0 e^{-\sigma} \omega_{\alpha\sigma}, \quad B_{mn} \equiv g^\mu_m g^\nu_n B_{\mu\nu} = B_0 e^{-\sigma} \omega_{mn}. \quad (162)$$

It is obvious (see (95) and (98)) that the electromagnetic field strength tensor satisfies the cyclic Maxwell system

$$B_{<mn;k>} = 0. \quad (163)$$

By substituting (162) in the expression (155) we find that the electromagnetic induction tensor is to be defined as follows:

$$H_{mn} = e^{3\sigma} B_{mn}, \quad (164)$$

and the constant  $B_0$  can be chosen as

$$B_0 = \pm \sqrt{\frac{8\pi}{\kappa_0}}. \quad (165)$$

In this case the electromagnetic part of the energy-momentum tensor  $\overset{4}{T}_{mn}$  (155) takes its usual form (in the Gaussian system of units):

$$E_{mn} = \frac{1}{4\pi} (B_{mk} H^k_n + \frac{1}{4} \overset{4}{g}_{mn} B_{jk} H^{jk}). \quad (166)$$

It is easy to see that the one-form

$$A_\mu \equiv B_0 S_0 P^\sigma_\mu \zeta_\sigma = B_0 S_0 P^\sigma_\mu \tau_{,\sigma} \quad (167)$$

has the following properties:

$$A_{\mu||\nu} - A_{\nu||\mu} = B_{\nu\mu} \quad \text{and} \quad A_{m,n} - A_{n,m} = B_{nm} \quad (A_m = g^\sigma_m A_\sigma). \quad (168)$$

Thus the orthogonal vector, projected into the hypersurface  $\mathcal{M}_4$  in an appropriate way, is the electromagnetic vector potential.

Now we are ready to expound the equation (91b). The result is

$$H^{mn}{}_{;n} = \frac{4\pi}{c} j^m, \quad (169)$$

where the abbreviations we used are given by

$$j^m = \frac{\kappa_0 B_0 c}{4\pi} \overset{4}{\theta}{}^m \quad \text{and} \quad \overset{4}{\theta}{}^m \equiv \overset{4}{g}_{mn} g^\mu_n \overset{5}{\theta}{}^\mu_\mu. \quad (170)$$

### 3.3.3 Field equation of the scalaric field $\sigma$

Using the Relations (150) and (90) we can rewrite the equation (91c) in the form

$$\sigma^{,m}{}_{;m} = \frac{\kappa_0}{8\pi} B_{mn} H^{mn} + \frac{2}{3} \kappa_0 \vartheta, \quad (171)$$

where the following definition was used:

$$\vartheta \equiv \overset{5}{\theta} e^{-\sigma} - \frac{1}{2} \overset{4}{g}{}^{mn} \overset{4}{\theta}_{mn}. \quad (172)$$

## 3.4 Version III

In the version III of PUFT the space-time metric  $\overset{4}{g}$  coincides with the on the hypersurface  $\mathcal{M}_4$  induced metric  $\tilde{g}$  (see (92)). This makes the projection formalism a little bit easier, because  $\epsilon = 0$ . On the contrary the 5-dimensional field equations (13) are more complicated. Further investigations have shown that the case  $\kappa_0 K_0 = -2$ , already mentioned above, is of a particular interest. Hence for the version III we find the following equation

$$\left. \begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \overset{5}{R} - \frac{1}{S} S_{,\mu;\nu} - \frac{2}{S^2} S_{,\mu} S_{,\nu} \\ &\quad - \frac{1}{S} S_{,\mu} S_{,\nu} \left( 4 S^{,\tau}{}_{;\tau} - \frac{6}{S} S_{,\tau} S^{,\tau} + \frac{3\lambda_0}{S} - \frac{S}{2} \overset{5}{R} \right) + \frac{1}{S} g_{\mu\nu} \left( S^{,\tau}{}_{;\tau} + \frac{\lambda_c}{S} \right) \\ \epsilon &= 0 \end{aligned} \right\}. \quad (173)$$

Following the above introduced procedure of deducing the field equations, one obtains the system of equations listed below [15].



### 3.4.1 Generalized gravitational field equation

$${}^4R_{mn} - \frac{1}{2} {}^4g_{mn} {}^4R + \frac{\lambda_0}{S_0^2} e^{-2\sigma} {}^4g_{mn} = \kappa_0 {}^4T_{mn}, \quad (174)$$

where  ${}^4T_{mn} = \theta_{mn} + E_{mn} + S_{mn}$  with

$$E_{mn} = \frac{1}{4\pi} (B_{mk} H^k_n + \frac{1}{4} {}^4g_{mn} B_{jk} H^{jk}), \quad S_{mn} = \frac{2}{\kappa_0} (\sigma_{,m} \sigma_{,n} - \frac{1}{2} {}^4g_{mn} \sigma_{,k} \sigma^{,k}) \quad (175)$$

holds.  ${}^4\theta_{mn} \equiv g^\mu_m g^\nu_n \tilde{\theta}^5_{\mu\nu}$  is the energy-momentum tensor of the substrate.

### 3.4.2 Generalized electromagnetic field equations (Maxwell equations)

$$a) H^{mn}{}_{;n} = \frac{4\pi}{c} j^m, \quad b) B_{[mn,k]} = 0, \quad c) H_{mn} = e^{2\sigma} B_{mn}, \quad (176)$$

where we used the abbreviations

$$a) B_{mn} \equiv g^\mu_m g^\nu_n B_{\mu\nu} = B_0 e^{-\sigma} \omega_{mn}, \quad b) j^m = \frac{\kappa_0 B_0 c}{4\pi} \theta^m e^\sigma \quad (177)$$

and  $\theta^m \equiv g^{mn} g^\mu_n \tilde{\theta}^5_\mu$ ,  $B_0 = \pm \sqrt{\frac{8\pi}{\kappa_0}}$ .

### 3.4.3 Field equation of the scalaric field $\sigma$

$$\sigma^{,m}{}_{;m} = -\frac{\kappa_0}{16\pi} B_{mn} H^{mn} - \frac{\kappa_0}{2} \vartheta - \frac{\lambda_0}{S_0^2} e^{-2\sigma} \quad \text{with} \quad \vartheta \equiv \tilde{\theta}^5 - g^{mn} \theta_{mn}. \quad (178)$$

## 4 Concluding Notes

Now let us summarize the basic ideas of the new geometrical approach to the axiomatics of Schmutzer's 5-dimensional Projective Unified Field Theory.

The mathematical basis for the 5-dimensional Projective Unified Field Theory forms the group of all 5-dimensional homogeneous coordinate transformations of degree one (19). The 5-dimensional geometry, constructed on this group, supposes the existence of a Killing vector field. The integral curves of this vector field form a Killing congruence (23) which is the basis of the projection formalism developed here. The angular velocity  $\omega_{\mu\nu}$  of this congruence is interpreted as the electromagnetic field strength tensor (see (162) and (176)). It is well known that, if  $\omega_{\mu\nu} = 0$  holds, a hypersurface, holonomic and orthogonal to the congruence exists. There are two possibilities to construct an axiomatics of PUFT: abandoning either **holonomicity** or **orthogonality**. The first of the two possibilities was investigated in detail in numerous papers by Schmutzer (see [6, 13, 15] and there quoted papers). The second possibility was considered in the present paper. In this case it is possible to say that the **holonomicity** of space-time and

the **non-orthogonality** of the given congruence with respect to the space-time hypersurface are embodied in the basis of the axiomatics offered here. In this way PUFT has got a new geometrical interpretation.

Inner curvature of space-time ( $\mathcal{M}_4$ ) identified with the hypersurface  $\tau = \text{const}$  (see page 7) describes the gravitation. The norm of the Killing vector field  $\xi$  (24) is connected with the new scalaric field  $\sigma$ :  $\sqrt{\xi^\nu \xi_\nu} = S_0 e^\sigma$ . The tensor of the angular velocity  $\omega_{\mu\nu}$  (30) of the Killing congruence (23) describes the electromagnetic field. Thus the orthogonal vector projected in an appropriate way onto the hypersurface  $\tau = \text{const}$  ( $\mathcal{M}_4$ ) is the electromagnetic vector potential (see (167) and (168)). The relation (167) implies that the electromagnetic potential vanishes if the hypersurface  $\mathcal{M}_4$  is orthogonal to congruence (23).

It is easy to show that physically the 4-dimensional field equations in the version II of PUFT differ slightly from the corresponding equations in the version III **physically** slightly. Here we won't dwell on this problem, therefore let us just remark that the cosmological term in the equation (1) can be accepted in the form  $\Lambda_{\mu\nu} = \lambda_0 e^{-\sigma} P_{\mu\nu}$  ( $\Lambda^\mu_{\nu;\mu} = 0$ ) as well. In this case additional terms containing a cosmological constant in the equations (154) and (171) take the form  $\lambda_0 e^{-2\sigma} \overset{4}{g}_{mn}$  and  $\frac{4}{3} \lambda_0 e^{-2\sigma}$ , respectively.

In conclusion let us emphasize that the axiomatics constructed here leads to the same 4-dimensional field equations which formerly were obtained by E.Schmutzer in a different way.

## 5 Appendix. Results of Application of PUFT

Since 1995 a series of papers by E.Schmutzer on a closed homogeneous isotropic cosmological model of the universe and on the influence of the expansion of such a model on cosmogony and astrophysics appeared ( see [15] where further literature is quoted) or are in press [16]. Let us mention some main results:

- In order to be in agreement with the equivalence principle the usual concept of mass is basically changed: mass depends on the cosmological scalaric field. Hence follows a considerable change of the cosmological situation at the start of the universe (fulfilling of certain aspects of Mach's principle).
- The big bang singularity does not exist. The "big start" (Urstart) of the universe begins softly and is (using a certain physically motivated choice of parameters) characterized by a kind of oscillations: expansion interrupted by small contractions.
- The cosmological scenario appears to be divided into a short repulsion (antigravitational) era (duration of 128 years) and a cosmologically long attraction era (age of the universe = 18 billions of years).

- The Hubble factor ("constant") is 75km/s Mpc.
  - Maxima and minima in the curves for the temporal behaviour of the cosmological mass density and the temperature could be interesting for the explanation of cosmogonic activities (birth of galaxies and stars).
  - The equation of motion of a body is in full agreement with the Einstein effects (periastron motion, deflection and frequency shift of electromagnetic waves).
  - Further consequences of the equation of motion are:
    - Time dependence of the "effective gravitational constant" with the present relative value:  $3.5 \cdot 10^{-11}$ /year.
    - For an orbiting body around a center: positive value of the angular (secular) acceleration, negative values of the time derivatives of the orbital radius (decrease), revolution period (decrease), excentricity (transition from elliptic to cyclic orbits).
    - Heat production in a moving body with application to the moon, planets, sun, galaxy etc. with remarkably interesting numerical results.
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